An Invitation to the Frobenius Problem

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Imagine that a shopkeeper has an infinite supply of coins of two different denominations: 5 cents and 7 cents. You wish to buy something which is priced 23 cents from a vending machine. You have money, but the vending machine accepts only 5 cent and 7 cent coins. Can the shopkeeper help? This problem about giving exact change can be easily put in a more mathematical setting as follows: does the equation $5x + 7y = 23$ admit a solution in nonnegative integers x and y? The answer to this question is No. What is more interesting is the role played by the number 23 in this problem; 23 is the largest number which cannot be so expressed. The mathematical generalization of this is easy to come up with. Given two positive integers a and b with $\gcd(a, b) = 1$, what is the largest integer n such that the equation $ax + by = n$ has no solution with $x, y \in \mathbb{Z}_{\geq 0}$. We use the notation $\mathtt{g}(a, b)$ to denote this largest integer, and use $\mathtt{n}(a, b)$ to denote the (finite) number of positive integers that cannot be expressed in the form $ax + by$ with $x, y \in \mathbb{Z}_{\geq 0}$. There is an obvious extension of this from coins of two denominations to one of k denominations, for any $k > 2$. The problem to determine the function $\mathtt{g}$ and $\mathtt{n}$ dates back to the 1880's, and is one of the most well known problems in additive Number theory, with a long and rich history. Whereas it is not difficult to show that $\mathtt{g}(a, b) = ab - a - b$ and that $\mathtt{n}(a, b) = \frac{1}{2}(a - 1)(b - 1)$ and has been known since the 1880's, there is no analogous result for more than two variables.

I will prove the two formulas stated above in several ways. I will then show how $\mathtt{g}$ and $\mathtt{n}$ can be determined in some special cases by using one or two basic tools and results. I will conclude by asking some important open questions regarding this problem. No background beyond a basic knowledge of modular arithmetic is required, and I plan to aim this at the interested and motivated undergraduate, hopefully without losing the interest of the more mature mathematician.

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About the Speaker. Amitabha Tripathi studied Mathematics at St. Stephen's College, University of Delhi, IIT Kanpur & State University of New York at Buffalo, from where he obtained a Ph.D. under the supervision of Prof. T. W. Cusick a little over three and a half decades ago. His research interests lie in elementary and combinatorial aspects of Number Theory, in combinatorial aspects of Graph Theory, and more recently, in Ramsey Theory and in Numerical Semigroups. Prof. Tripathi has been with the Department of Mathematics at IIT Delhi since July 1991, where in addition to teaching a wide variety of courses, he has mentored and helped guide research among undergraduate students that has led to several fruitful collaborations. Outside of Mathematics, he enjoys listening to old Hindi film songs, playing Table Tennis, solving crosswords and performing speed calculations, in that order.