Solved and Unsolved Problems in Number Theory

Amitabha Tripathi

Vaidya Saolapurkar Chair Professor, Department of Mathematics, Indian Institute of Technology Delhi

e-mail: atripath@maths.iitd.ac.in

Number Theory may be loosely defined to be the study of the natural numbers, $1, 2, 3, \ldots$. As the German mathematician L. Kronecker (1823 – 1891) famously remarked "*Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk*", which translates to "God made the integers, all else is the work of man". The natural numbers, together with their negatives and 0 constitute the set of integers, and the study of their properties has provided generations of mathematicians an unending supply of challenging problems. Prime numbers, like $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, \ldots$, are positive integers that cannot be divided evenly by positive integers other than 1 and itself. The ancient Greek mathematician Euclid (c. 300 BC) was already aware that there is an unending supply of prime numbers. Arithmetic, as Number Theory was then called, together with Geometry flourished during the times of the ancient Greeks. Prime numbers are the building blocks for the integers – every integer can be expressed as a product of prime numbers in an essentially unique manner. There are many fascinating problems concerning prime numbers, some of which have been answered and some still seek to be resolved. Two of the most famous unsolved problems concerning primes are the `Goldbach Conjecture` ("Every even integer greater than 2 is a sum of two primes" – C. Goldbach, 1742) and the `Twin Prime Conjecture` ("There are infinite many pairs of primes that differ by 2"). One of the crowning glories of mathematics in the 19th century was the resolution of a question in regard to the distribution of primes – how many primes can we expect up to a "large" positive integer N as a proportion to N? Using methods which were hitherto unthought of – tools from Complex Function theory – two French mathematicians J. Hadamard (1865 – 1963) and C.-J. de la Vallée-Poussin (1866 – 1962) independently proved that about $1/\ln N$ of the numbers up to N are prime.

As the title of my presentation suggestions, I will speak on a wide variety of important problems in Number Theory. `Fermat's Last Theorem` (FLT) had attained the status of one of the most difficult problems to solve in Mathematics, partly due to the fact that many great mathematician had tried and failed to solve it. But its importance was equally due to the fact that large bodies of mathematical thought were created in order to understand its proof. When the English mathematician A. Wiles (b. 1953) finally solved the 357 year old problem in 1994, the proof was hailed as `The Proof of the Century`. `Riemann's Hypothesis` (RH) is a lot more technical in nature, and has therefore not been as attractive to the amateur mathematician as was the FLT. The problem was enunciated more than a century and a half ago by G. B. H. Riemann in 1859; however, despite a great deal of effort by mathematicians of the 20th and 21st century, a solution seems out of range just now.

Number Theory undoubtedly attracts more amateurs than any other branch of modern Mathematics. This is because there remain many problems in Number Theory that can be understood by an enthusiastic school child. My presentation will aim to introduce a number of such problems that have attracted the attention of the amateur and the professional mathematician alike.